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Estimation Problem for Impulsive Control Systems under Ellipsoidal State Bounds and with Cone Constraint on the Control

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Abstract. The paper deals with the state estimation problem for the linear control system containing impulsive control terms (or measures). The problem is studied here under uncertainty conditions when the initial system state is unknown but bounded, with given bound. It is assumed also that the system states should belong to the given ellipsoid in the state space. So the main problem of estimating the reachable set of the control system is studied here under more complicated assumption related to the case of state constraints. It is assumed additionally that impulsive controls in the dynamical system must belong to the intersection of a special cone with a generalized ellipsoid both taken in the space of functions of bounded variation. The last constraint is motivated by problems of impulsive control theory and by models from applied areas when not every direction of control impulses is acceptable in the system. We present here the state estimation algorithms that use the special structure of the control system and take into account additional restrictions on states and controls. The algorithms are based on ellipsoidal techniques for estimating the trajectory tubes of uncertain dynamical systems. Numerical simulation results related to proposed procedures are also given.

Keywords: Impulsive control, reachable sets, estimation.
PACS: 02.30.Yy

INTRODUCTION

This work relates to one of the important parts of the dynamic optimization — impulse control, where we study the dynamic processes with discontinuous trajectories and generalized controls impulsive type. For a wide range of control problems you need to know a reachable set. The reachable set is the set of states of the phase space where the phase point can be moved from the initial state (or a set of initial conditions) for a given time by some admissible control [8, 9].

In many applied problems, exact finding of the reachable set may be very difficult. Often instead of exact reachable set we may consider its approximating figure of simple structure, e.g. of ellipsoidal form. For the formation of an optimal control problem under study rougher, replacing the exact reachable set his assessment of the integration (internal and external). One possible method of constructing the external and internal approximations of reachable sets for is a method of ellipsoidal approximations may be found in [2, 11].

In this paper we consider the problem of estimation of reachable sets of the impulsive control system [8, 9] with the uncertainty in initial data. We assume that impulsive

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controls in the linear dynamical system have to belong to the intersection of a special cone with a generalized ellipsoid both taken in the space of functions of bounded variation. In particular, under such restriction vectors of impulsive jumps of admissible controls are taken from a given finite-dimensional ellipsoid intersected with the cone. We study here the special case of such restricting cone when only nonnegative (in each coordinate) control measures are allowed in the system [1, 5, 9].

We consider the problem under additional constraint on the system state, namely we assume also that the system states should belong to the given ellipsoid in the state space.

We present here the state estimation algorithms that use the special structure of the control system and take into account additional constraints [2, 11] on states and controls. The algorithms are based on ellipsoidal techniques for estimating the trajectory tubes of uncertain dynamical systems. Numerical simulation results related to proposed procedures are also given.

PROBLEM FORMULATION

Let $\mathbf{R}^n$ denote the n-dimensional Euclidean space. Denote by $\mathbf{R}^{n\times n}$ the set of all $n\times n$ -matrix.

Consider a dynamic control system described by a differential equation with impulsive control $u(\cdot)$:

$$dx(t) = A(t)x(t)dt + du(t), \;\; x(t_0-0) = x_0, \;\; t\in[t_0,T], \tag{1}$$

or in the integral form

$$x(t) = x(t;u(\cdot),x_0) = \Phi(t,t_0)x_0 + \int\limits_{t_0}^{t}\Phi(t,\tau)du(\tau). \tag{2}$$

Here we assume that $A(t)$ is a continuous $n\times n$ – matrix function, $\Phi(t)$ is the Cauchy matrix, $\Phi(t,\tau)=\Phi(t)\Phi^{-1}(\tau)$, $u(\cdot)\in\mathbf{V}_p^n$ where $\mathbf{V}_p^n$ $(1\le p<\infty)$ means the space of n-vector functions $u(\cdot)$ such that $u(t)$ is continuous from the right on $[t_0,T)$ with $u(-0)=0$ and

$$V_p[u(\cdot)] = \sup_{\{t_i\}}\sum_{i=1}^{k}\|u(t_i)-u(t_{i-1})\|_p<\infty, \qquad \|u\|_p = \left(\sum_{i=1}^{n}|u_i|^p\right)^{\frac{1}{p}},$$

where $u=(u_1,\ldots,u_n)$ and $t_i:\; t_0<\ldots<t_k=T$.

Denote $\mathbf{C}_q^n$ the space of continuous n-vector functions $y(\cdot)$ with the norm

$$\|y(\cdot)\|_{\infty,q} = \max_{t_0\le t\le T}\|y(t)\|_q.$$

It is well known [3] that the space $\mathbf{V}_p^n=\mathbf{C}_q^{n*}$ where $p=1$ if $q=\infty$, $p=\infty$ if $q=1$ and $1<p<\infty$ if $q=(1-p^{-1})^{-1}$.

Denote $E(y,\,Y)=\left\{x\in\mathbf{R}^n:(x-y)'Y^{-1}(x-y)\le 1\right\}$ be the ellipsoid in $\mathbf{R}^n$ with center y and the symmetric positive definite matrix $Y\in\mathbf{R}^{n\times n}$, and let $I\in\mathbf{R}^{n\times n}$ be the identity matrix.

Let $E_0 = E(0, Q_0^{-1})$ be the ellipsoid in $\mathbf{R}^n$ with $Q_0 \in \mathbf{R}^{n\times n}$ being a given symmetric positive definite matrix. Consider the so-called generalized "ellipsoid" [4, 15] E in $\mathbf{C}_q^n$:

$$E = \{y(\cdot) \in \mathbf{C}_q^n \mid \ y(t) \in E_0 \ \ \forall\, t \in [t_0, T]\} \tag{3}$$

and its conjugate ellipsoid [13] $E^* \subset \mathbf{V}_p^n$,

$$E^* = \{u(\cdot) \in \mathbf{V}_p^n \mid \int_{t_0}^{T} y'(t)du(t) \leq 1 \ \forall y(\cdot) \in E\}. \tag{4}$$

Denote

$$K_0 = R_+^n = \{u \in \mathbf{R}^n \mid u = (u_1, \ldots, u_n),\ u_i \geq 0,\ i = 1, \ldots, n\},$$
$$K = \{y(\cdot) \in \mathbf{C}_q^n \mid y(t) \in K_0 \ \forall\, t \in [t_0, T]\}$$

and take the conjugate cone [13]

$$K^* = \{u(\cdot) \in \mathbf{V}_p^n \mid \int_{t_0}^{T} y'(t)du(t) \geq 0 \ \ \forall y(\cdot) \in K\}. \tag{5}$$

Definition 1. *The function $u(\cdot) \in \mathbf{V}_p^n$ will be called the admissible control if $u(\cdot) \in U = E^* \cap K^*$.*

Remark 1. Let $u(\cdot)$ be a piecewise constant function on $[t_0, T]$ with discontinuity instants $\{t_i\}$ and with jump vectors

$$\begin{aligned} \Delta u = u(t_{i+1}) - u(t_i) \in E(0, Q_0) \cap K_0 = \\ = \{z \in \mathbf{R}^n |\ z' Q_0^{-1} z \leq 1,\ z = (z_1, \ldots, z_n),\ z_i \geq 0,\ i = 1, \ldots, n\}. \end{aligned} \tag{6}$$

Then $u(\cdot)$ is admissible.

We will assume also that the initial state x_0 to the system (1) is unknown but bounded with given ellipsoidal bound,

$$x_0 \in X_0 = E(r, R), \tag{7}$$

where $R \in \mathbf{R}^{n\times n}$ is a symmetric positive definite matrix and $r \in \mathbf{R}^n$.

Let's in addition assume that the current state $x(t)$ of the system (1) is constrained by the ellipsoidal restriction

$$x(t) \in E(y_0, D) \in \mathbf{R}^n, \quad t_0 \leq t \leq T, \tag{8}$$

where $D \in \mathbf{R}^{n\times n}$ is a symmetric positive definite matrix and $y_0 \in \mathbf{R}^n$.

Denote

$$X(t; U, X_0) = \bigcup_{x_0 \in X_0} \bigcup_{u \in U} \{x(t; u(\cdot), x_0)\}. \tag{9}$$

Definition 2. *The set $X(T; U, X_0)$ (9) is called the reachable set of the impulsive differential system* (1) *from the initial set X_0* (7) *at the instant T under controls $u(\cdot) \in U$ and ellipsoidal state bounds* (8).

So the main problem of the paper is to find the external estimates of ellipsoidal type for the reachable set $X(T; U, X_0)$ basing on the special ellipsoidal structure of sets X_0 and U and ellipsoidal state constraint (8).

MAIN RESULTS

For a set $A \subset \mathbf{R}^n$ we denote its closed convex hull [13] as $\overline{\mathrm{co}}A$.

The follow theorem is true.

Theorem 1. *The equality holds*

$$\begin{gathered} X(T;U,E(r,R)) = E(r^*,R^*) + X(T;U,0), \\ X(T;U,0) = \overline{\mathrm{co}}\Big(\bigcup_{\tau\in[t_0,T]} \Phi(T,\tau)(E(0,Q_0)\cap K_0)\Big), \\ r^* = \Phi(T,t_0)r, \quad R^* = \Phi(T,t_0)R(\Phi(T,t_0))'. \end{gathered} \tag{10}$$

Remark 2. Assume that we find an ellipsoid E^+ such that $X(T;U,\{0\}) \subseteq E^+$, where $X(T;U,\{0\})$ is defined in (9) for $X_0 = \{0\}$. Then applying well-known formulas for calculating the external ellipsoidal estimate for the sum of two ellipsoids $E(r^*,R^*)$ and E^+ [2, 11] we can find the resulting external ellipsoid for $X(T;U,X_0))$ in (10). So the main difficulty now is in constructing the outer ellipsoidal estimate for $X(T;U\{0\})$.

Therefore we apply the techniques of the ellipsoidal calculus to find the external estimates for $X(T;U,\{0\})$. The idea of constructing the external estimates for $X(T;U,\{0\})$ is based on results of ellipsoidal calculus [2, 11] and on the new procedures of external approximation of a closed convex hull of the union of a variety of some ellipsoids [4, 12, 14, 15].

The main problem will be solved in several steps. The resulting algorithm will be done later.

First, we find the ellipsoid containing the intersection $E(0,Q_0)\cap K_0$.

Theorem 2. [6] *For any $i = 1,\ldots,n$ the following inclusion holds*

$$\{z\in\mathbf{R}^n \,|\, z'Q_0^{-1}z \le 1,\ z=(z_1,\ldots,z_n),\ z_i\ge 0\} \subseteq E(a_i,Q_i) \tag{11}$$

with

$$\begin{gathered} a_i = (n+1)^{-1}((Q_0)_{ii})^{-1/2}Q_0^{(i)}, \\ Q_i = (n^{-2}(n^2-1)Q_0^{-1} + 2(n+1)n^{-2}((Q_0)_{ii})^{-1}V^{(i)})^{-1}, \end{gathered} \tag{12}$$

where $Q_0^{(i)}$ and $(Q_0)_{ii}$ denote, respectively, the i-column and (i,i)-element of $n\times n$-matrix Q_0, $V^{(i)}$ is the $n\times n$-matrix with elements $V_{sk}^{(i)}$ $(1\le s,k\le n)$ such that $V_{sk}^{(i)} = 1$ if $s=k=i$, otherwise $V_{sk}^{(i)} = 0$.

Directly from Theorem 3 we come to the estimate.

Corollary 1. *The following external estimate is true*

$$E(0,Q_0)\cap K_0 \subseteq \bigcap_{1\le\ i\le n} E(a_i,Q_i), \tag{13}$$

where a_i, Q_i are defined by formulas (12).

Theorem 3. [6] *For any parameter vector* $\alpha = (\alpha_1, \ldots, \alpha_n)$ *with* $\alpha_i \geq 0$ *and* $\sum\limits_{i=1}^{n} \alpha_i = 1$ *the following inclusion holds*

$$E(0, Q_0) \cap K_0 \subseteq E(a_\alpha, Q_\alpha) \tag{14}$$

where

$$a_\alpha = \Big(\sum_{i=1}^{n} \alpha_i Q_i^{-1} \Big)^{-1} \cdot \sum_{i=1}^{n} \alpha_i Q_i^{-1} a_i, \qquad Q_\alpha = (1 - h^2(\alpha)) \Big(\sum_{i=1}^{n} \alpha_i Q_i^{-1} \Big)^{-1}, \tag{15}$$

$$h^2(\alpha) = \sum_{i=1}^{n} \alpha_i a_i' Q_i^{-1} a_i - \Big(\sum_{i=1}^{n} \alpha_i Q_i^{-1} a_i \Big)' \Big(\sum_{i=1}^{n} \alpha_i Q_i^{-1} \Big)^{-1} \Big(\sum_{i=1}^{n} \alpha_i Q_i^{-1} a_i \Big), \tag{16}$$

where a_i*,* Q_i *are defined in* (12).

Remark 3. The estimate (14) is valid for any convex combination $\{\alpha_i\}$ ($\alpha_i \geq 0$, $\sum \alpha_i = 1$, $i = 1, \ldots, n$). To simplify the calculations we may take $\alpha_i = 1/n$ for all $i = 1, \ldots, n$.

Consider impulsive control system

$$dx(t) = A(t)x(t)dt + du(t), \ \ x(t_0 - 0) = x_0 \in X_0 = E(r, R), \ \ t \in [t_0, T], \tag{17}$$

with constraints

$$u(t) \in \tilde{U} = \tilde{E}_\alpha^* \in \mathbf{V}_p^n, \tag{18}$$

where

$$\tilde{E}_\alpha^* = \{ u(\cdot) \in \mathbf{V}_p^m \mid \int_{t_0}^{T} y'(t) du(t) \leq 1 \ \forall y(\cdot) \in \tilde{E}_\alpha \},$$

$$\tilde{E}_\alpha = \{ y(\cdot) \in \mathbf{C}_q^n \mid \ y(t) \in E_\alpha^0 = E(a_\alpha, Q_\alpha) \ \ \forall \ t \in [t_0, T] \}$$

and the same state constraints in (8)

$$x(t) \in E(y_0, D) \in \mathbf{R}^n, \quad t_0 \leq t \leq T, \tag{19}$$

where parameters a_α, Q_α are defined in (15)–(16).

Consider the follow auxiliary problem.

Auxiliary Problem AP. Find the ellipsoid $E_\alpha^+(a_\alpha^+, Q_\alpha^+)$ such that $X(t; \tilde{U}, X_0) \subseteq E_\alpha^+(a_\alpha^+, Q_\alpha^+)$, where $X(t; \tilde{U}, X_0)$ is the reachable set of system (17)–(19).

Theorem 4. *The following inclusion holds*

$$X(t; U, X_0) \subseteq X(t; \tilde{U}, X_0). \tag{20}$$

Proof. The proof of this theorem follows from the definition and properties of the set $\tilde{U}$ (19) .

Now we consider the main scheme of construction the external ellipsoidal estimates of reachable sets $X(t;\tilde{U},X_0)$ of the system (17)–(19). This scheme is based on the idea [10] of transformation of the problem with state constraints into the problem without coordinates restrictions. This transformation is made by including the special matrix of parameters in the dynamical system.

Consider the following auxiliary system [7]

$$\begin{aligned} dz(t) = ((A(t)-L)z(t)+v(t))dt+du(t), \qquad t_0 \le t \le T, \qquad u(t) \in \tilde{U}, \\ v(t) \in LE(y_0,D), \qquad z(t_0-0) = z_0 \in X_0 = E(r,R), \qquad L \in \mathbf{R}^{n\times n}. \end{aligned} \tag{21}$$

Let $z[\cdot] = z(\cdot;u,z_0,L)$ be a solution of system (21) for $t \in [t_0,T]$ under initial condition $z(t_0-0) = z_0 \in X_0$ for a given admissible functions $u \in \tilde{U}$ and L.

Denote

$$Z(\cdot;\tilde{U},X_0,L) \subseteq \bigcup \{z(\cdot;u,z_0,L) \mid z_0 \in X_0, \quad u(t) \in \tilde{U}\}.$$

Theorem 5. [7] *The following inclusion holds*

$$X(T;\tilde{U},X_0) \subseteq \bigcap_L Z(T;\tilde{U},X_0,L), \tag{22}$$

where $Z(T;\tilde{U},X_0,L)$ is a reachable set of linear impulsive control system (21).

Note that each estimate of $Z(T;\tilde{U},X_0,L)$ gives the outer ellipsoidal approximation of the set $X(T;\tilde{U},X_0)$. A more accurate estimate is obtained by the intersection of the ellipsoids generated by different values of the parameter matrix L.

Each estimate of $Z(T;\tilde{U},X_0,L)$ gives the external ellipsoidal approximation of the set $X(T;\tilde{U},X_0)$, and hence the estimate the reachable set $X(t;U,X_0)$ its follows from (20).

Now our next aim is to apply the techniques of the ellipsoidal calculus to find the estimates for the $Z(T;\tilde{U},X_0,L)$.

Consider also two auxiliary problems.

Auxiliary Problem AP1. For a given $L \in \mathbf{R}^{n\times n}$ find an external ellipsoidal estimate $E_L(a_1^+,\mathrm{Q}_1^+)$ of the reachable set $Z_1(T;\tilde{U},X_0,L)$ of impulsive system

$$\begin{aligned} dz(t) = (A(t)-L)z(t)dt+du(t), \qquad t_0 \le t \le T, \qquad u(t) \in \tilde{U}, \\ z(t_0-0) = z_0 \in X_0 = E(r,R), \qquad L \in \mathbf{R}^{n\times n}. \end{aligned} \tag{23}$$

The solution of the Auxiliary Problem AP1 was considered in [14].In these papers in order to find ellipsoidal estimates we used the fact that the reachable set $Z_1(T;\tilde{U},X_0,L)$ of system (23) has the following forms:

$$Z_1(T;\tilde{U},X_0,L) = \bigcup_{x_0\in X_0}\bigcup_{u\in\tilde{U}}\{\Phi(T,t_0;L)x_0 + \int_{t_0}^{T}\Phi(T,\tau;L)du(\tau)\}, \tag{24}$$

where $\Phi(t,\tau;L) = \Phi(t;L)\Phi^{-1}(\tau;L)$, $\Phi(t;L)$ is the Cauchy matrix of system (23), and that for any $\varepsilon{>}0$ there exist $\delta{>}0$ and finite set $T_\delta{\subset}[0,T]$ such that the inclusions are true

$$X_0 + \overline{\mathrm{co}}(\bigcup_{\tau\in T_\delta} E(0,Q_\tau)) \subset Z_1(T;\tilde{U},X_0,L) \subset X_0 + \overline{\mathrm{co}}(\bigcup_{\tau\in T_\delta} E(0,Q_\tau)) + \varepsilon S,$$

where $Q_\tau = \Phi(T,\tau;L)Q_0\Phi'(T,\tau;L)$. So, in the case of Auxiliary Problem AP1 we need to construct the external ellipsoidal estimates for the convex hull of the union of a family of ellipsoids. This problem does not arise in the case of estimating states of dynamic systems with controls of classical type [9] and is new for estimation problems in impulsive systems [14].

Auxiliary Problem AP2. Find an external ellipsoidal estimates $E_L(a_2^+,Q_2^+)$ for the reachable set $Z_2(T,LE(y_0,D),\{0\},L)$ of system

$$\begin{gathered} dz(t) = (A(t)-L)z(t)dt + v(t)dt, \qquad t_0 \le t \le T, \\ z(t_0-0) = 0, \qquad v(t) \in LE(y_0,D). \end{gathered} \tag{25}$$

In this case the reachable set $Z_2(T,LE(y_0,D),\{0\},L)$ of system (25) has the form:

$$Z_2(T,LE(y_0,D),\{0\},L) = \bigcup_{v\in LE(y_0,D)} \int_{t_0}^{T} \Phi(T,\tau;L)v(\tau)d\tau. \tag{26}$$

The Auxiliary Problem AP2 is the classical problem. The solution of this problem may be found in the following monographes [2, 11].

The following simple lemma is true.

Lemma. [16] *The reachable set $Z(T;\tilde{U},X_0,L)$ of system (21) has the form*

$$Z(T;\tilde{U},X_0,L) = Z_1(T;\tilde{U},X_0,L) + Z_2(T;LE(y_0,D),\{0\},L). \tag{27}$$

In order to find parameters a^+, Q^+ of the ellipsoid $E_L^+ = E_L(a^+,Q^+)$ that contains the sum of ellipsoids $E_L(a_1^+,\mathrm{Q}_1^+)$ and $E_L(a_2^+,\mathrm{Q}_2^+)$ we use formulas [2, 11]:

$$E_L(a_1^+,\mathrm{Q}_1^+) + E_L(a_2^+,Q_2^+) \subset E_L^+ = E_L(a^+,Q^+),$$

$$a^+ = a_1^+ + a_2^+, \qquad Q^+ = (1/p+1)Q_1^+ + (p+1)Q_2^+,$$

where p is the unique positive root the following equation

$$\sum_{j=1}^{n} \frac{1}{p+\lambda_j} = \frac{n}{p(p+1)},$$

λ_j are the roots of the equation $\det(Q_1^+ - \lambda Q_2^+) = 0$.

From (27) that we have the inclusion

$$Z(T;\tilde{U},X_0,L) = Z_1(T;\tilde{U},X_0,L) + Z_2(T,LE(y_0,D),\{0\},L) \subset E_L^+ = E_L(a^+,Q^+). \tag{28}$$

Combining results of solution of the Auxiliary Problems AP, AP1 and AP2, of above Lemma and formulas (28) we may formulated the follow result.

Theorem 6. *The follows inclusions are true*

$$X(T;U,X_0) \subset E_L^+ = E_L(a^+,Q^+). \tag{29}$$

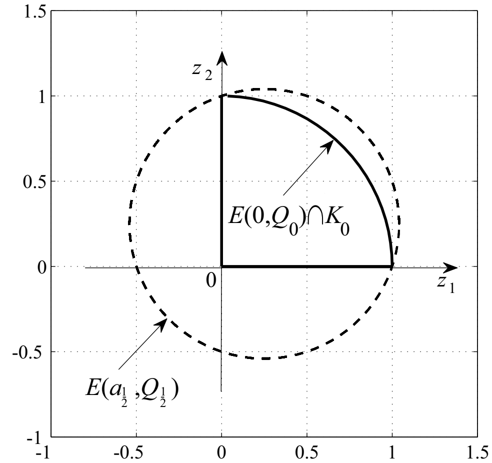

FIGURE 1. Ellipsoidal estimate of the $E(0,Q_0)\cap K_0$.

Thus, we obtain an algorithm of constructing an ellipsoidal estimate of the reachable set $X(T;U,X_0)$.

Algorithm.

Step 1. Fix the parameter vector $\alpha = (\alpha_1,\ldots,\alpha_n)$ with $\alpha_i \geq 0$ and $\sum_{i=1}^{n} \alpha_i = 1$. Find the ellipsoid $E(a_\alpha, Q_\alpha)$ containing the intersection $E(0,Q_0)\cap K_0$ by the formulas (12), (15)–(16). We turn to the Auxiliary Problem AP of ellipsoidal estimation of the reachable set $X(t;\tilde{U},X_0)$ of linear impulsive system for $\tilde{U}$ described in (18).

Step 2. Fix an arbitrary $L \in \mathbf{R}^{n\times n}$. We form the auxiliary system of the type (21). Each estimate of $Z(T;\tilde{U},X_0,L)$ gives the outer ellipsoidal approximation of the set $X(T;\tilde{U},X_0)$. So now we have to construct an ellipsoidal estimate of sets $Z(T;\tilde{U},X_0,L)$.

Step 3. Subdivide the Auxiliary Problem AP into Auxiliary Problem AP1 and Auxiliary Problem AP2.

a. Find ellipsoidal estimates $E_L(a_1^+,\mathrm{Q}_1^+)$ for the reachable set $Z_1(T;\tilde{U},X_0,L)$ of impulsive system (23) (for Auxiliary Problem AP1).

b. Find an external ellipsoidal estimates $E_L(a_2^+,Q_2^+)$ for the reachable set $Z_2(T,LE(y_0,D),\{0\},L)$ of system (25) (for Auxiliary Problem AP2).

Step 4. Find the ellipsoid E_L^+ (29) such that

$$E_L(a_1^+,\mathrm{Q}_1^+) + E_L(a_2^+,Q_2^+) \subset E_L^+.$$

Therefore, we will have the desired ellipsoidal estimates of reachable sets $X(T;U,X_0)$ of system (1) from the initial set X_0 (7) at the instant T under controls $u(\cdot)\in U$ and ellipsoidal state bounds (8).

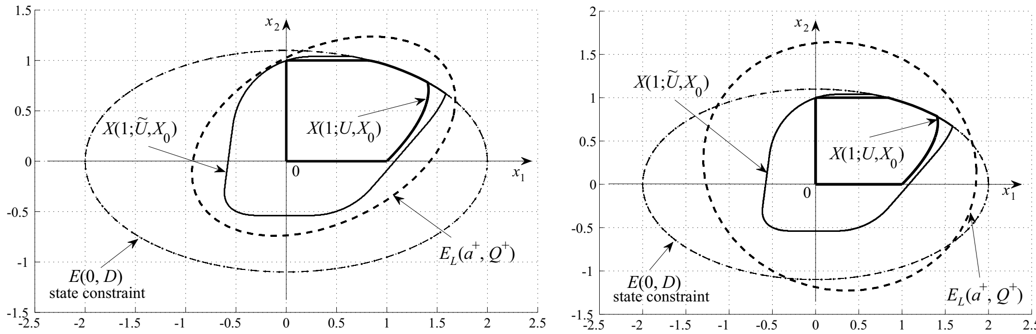

FIGURE 2. Ellipsoidal estimate of reachable set $X(1;U,X_0)$ for $L = L_1$ and $L = L_2$ respectively.

NUMERICAL SIMULATION: EXAMPLE

Consider the following control system:

$$\left\{ \begin{array}{lcl} dx_1(t) & = & x_2(t)dt + du_1(t), \\ dx_2(t) & = & du_2(t), \end{array} \right. \qquad t \in [0;1]. \tag{30}$$

We assume that $X_0 = \{0\}$ and the set $U = E^* \cap K^*$, where E^* is generated by of the ellipsoid

$$E(0,Q_0) = \left\{ l \in R^2 \mid \;\; l'Q_0 l \le 1 \right\}, \qquad Q_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and K^* is generated by of the $K_0 = \{u \in R^2 \mid u = (u_1, u_2),\ u_1 \ge 0,\ u_2 \ge 0\}$.

Let's in addition assume that the current state $x(t)$ of the system (1) is constrained by the ellipsoidal restriction

$$x(t) \in E(0,D) \in \mathbf{R}^n, \qquad D = \begin{pmatrix} 4 & 0 \\ 0 & 1.21 \end{pmatrix}, \qquad t \in [0;1].$$

The ellipsoid $E(a_{\frac{1}{2}}, Q_{\frac{1}{2}})$ containing the intersection $E(0,Q_0) \cap K_0$ given at the Fig. 1. Here we have $E(0,Q_0) \cap K_0 \subseteq E(a_{\frac{1}{2}}, Q_{\frac{1}{2}})$, where parameters $a_{\frac{1}{2}}$, $Q_{\frac{1}{2}}$ are found according to formulas (15)–(16). The exact reachable set $X(1;U,X_0)$ from initial set X_0 under restriction on the impulsive control function from $u \in U$ is presented at Fig. 2. Applying the numerical algorithms of contracting the external ellipsoidal estimates we found ellipsoids $E^+_{L_1} = E(a^+_{L_1}, Q^+_{L_1})$ with the matrix parameter $L_1 = \begin{pmatrix} 0 & 0.02 \\ 0.003 & 0.01 \end{pmatrix}$ and $E^+_{L_2} = E(a^+_{L_2}, Q^+_{L_2})$with the matrix parameter $L_2 = \begin{pmatrix} 0 & 0.7 \\ 0.3 & 0.0001 \end{pmatrix}$. The sets $E^+_{L_1}$ and $E^+_{L_2}$ and the exact reachable sets $X(1;U,\{0\})$ $X(1;U_1,\{0\})$ from initial set X_0 under restrictions on impulsive function U and U_1 are shown at the Fig. 2.

CONCLUSION

We presented here the approach that allows to find the external ellipsoidal estimate of the reachable sets of linear impulsive control systems with uncertainty in the initial data and in the presence of ellipsoidal state constraints. Impulsive controls are constrained by the intersection of a special cone with a generalized ellipsoid both taken in the space of functions of bounded variation. The example that illustrates the techniques discussed in the paper is also given.

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